Notes by
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Graph Theory:

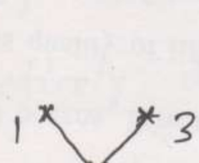
Definition: A Graph $G=(V,E)$ consists of a vertex set $V \equiv V(G)$ and an edge set $E \equiv E(G)$.

Example: Let $V=\{1,2\}$ and $E=\phi$. Then the graph $G=(V,E)$ may be drawn as $\overset{1}{\cdot}\ \overset{2}{\cdot}$

* The vertex set is usually a (finite) discrete set and if $V=\{v_1, v_2, \cdots, v_n\}$, an edge usually represented as (v_i, v_j) for some $i,j \in \{1,2,\cdots,n\}$.

* If (v_i, v_j) is an edge in E, we connect v_i & v_j by a line in $G \equiv (V,E)$.

Example: ①

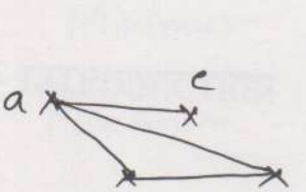


$V=\{1,2,3\}$

$E=\{\{1,2\},\{2,3\}\}$

$\{2,2\}$

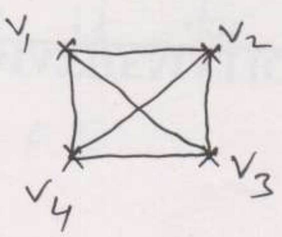
②



$V=\{a,b,c,d\}$

$E=\{\{a,c\},\{a,b\},\{a,d\},\{b,d\}\}$

③



$V=\{v_1,v_2,v_3,v_4\}$

$E=\{(v_i,v_j) : i \neq j,\ i,j \in \{1,2,3,4\}\}$

Definition: (Loop): A Loop is an edge whose endpoints are equal.

Example: Edges of the form (v,v).

Definition: (Multiple edge): Multiple edges are edges having the same pair of vertices.

Example: $(v_1,v_2), (v_1,v_2)$ are two edges between v_1 & v_2.

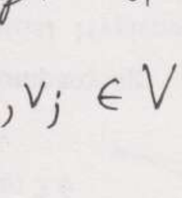
Definition: A Simple Graph is a graph having no loops or multiple edges.

* A null graph is $G \equiv (V,E)$ where $V = \phi = E$.
* The number of elements in V and E, denoted by $|V|$ and $|E|$ respectively are called the size and order of the graph $G=(V,E)$.

Note! Minimum possible order is 0 (for null graph)

Maximum " " " $\frac{n(n-1)}{2}$ (for complete graph)

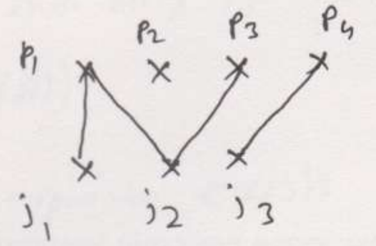
Complete Graph: A graph $G=(V,E)$ is said to be complete if for any $v_i, v_j \in V$ with $i \neq j$, we have $(v_i, v_j) \in E$.



Definition: (Bipartite Graph):

A graph $G \equiv (V, E)$ is called a Bipartite graph if the vertex set V can be expressed as $V_1 \cup V_2$ such that $V_1 \cap V_2 = \phi$ and for any $u, v \in V_i$ $(i=1,2)$, $(u,v) \notin E$.

Example:



$V = \{p_1, p_2, p_3, p_4, j_1, j_2, j_3\} = V_1 \cup V_2$

where $V_1 = \{p_1, p_2, p_3\}$ & $V_2 = \{j_1, j_2, j_3\}$.

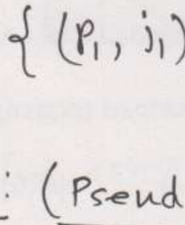
$E = \{(p_1, j_1), (p_1, j_2), (p_3, j_2), (p_4, j_3)\}$.

Definition: (Pseudo-graph):

A graph that contains loops as well as multiple edges between vertices is called a Pseudo-Graph.

Definition (Complete bibartite graph):

A bipartite graph in which all possible edges are present.



$K_{1,4}$

$K_{3,2}$

Graph Isomorphism:

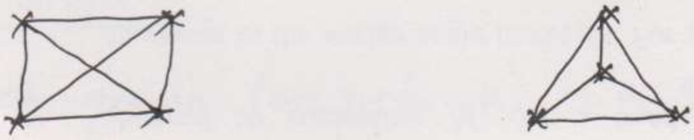
An isomorphism of graphs G and H is a bijection between the vertex sets ~~and~~ of G and H, say,

$$f: V(G) \rightarrow V(H)$$

such that any two vertices u and v form an edge in $G = (V(G), E(G))$ if and only if $f(u)$ and $f(v)$ form an edge in $H = (V(H), E(H))$.

If an isomorphism exists between two graphs, then the graphs are called Isomorphic and denoted as $G \simeq H$.

Example:

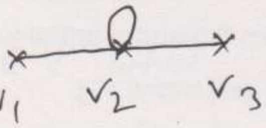


Two complete graphs on 4 vertices are isomorphic.

Representation of graphs by matrices:

Adjacency matrix:

Let $G = (V, E)$ be a graph and suppose $V = \{v_1, v_2, \cdots, v_n\}$. The adjacency matrix of the graph $G = (V, E)$ has its ij-th entry equal to 1 if $(v_i, v_j) \in E$ and 0 otherwise.



$G = (V, E)$

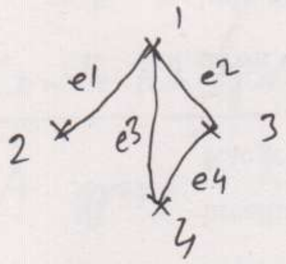
$$A(G) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Incidence matrix:

The incidence matrix of a graph $G=(V,E)$ is an $n \times m$ matrix B where $n = |V|$ and $m = |E|$, such that

$$B_{ij} = \begin{cases} 1 & \text{if vertex } v_i \text{ is incident with edge } e_j, \\ 0 & \text{o.w.} \end{cases}$$

Example:



$$\begin{array}{c|cccc} & e1 & e2 & e3 & e4 \\ \hline 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 1 \\ 4 & 0 & 0 & 1 & 1 \end{array} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Weighted Graph:

A weighted graph (or network) is a graph in which a number is assigned to each edge.

Such weights might represent costs, lengths, capacities etc.

Regular Graph:

A regular graph is a graph where each vertex has the same degree. A regular graph with vertices degree $k\ (\in \mathbb{N})$ is called a k-regular graph.

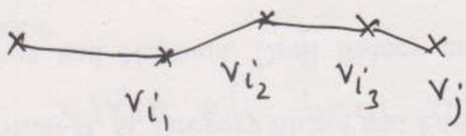
Degree of a vertex in a graph:

Let $v \in V$ where $G=(V,E)$ is a graph. Then the degree of the vertex v is the cardinality of the set,

$$\{(v, v_i) \in E(G) \mid v_i \in V(G)\}.$$

Path in a graph:

In a graph $G=(V,E)$, we say there is a path from v_i to v_j ($v_i, v_j \in V(G)$) if $\exists$ vertices $v_{i_1}, v_{i_2}, \dots, v_{i_k} \in V(G)$ such that $(v_i, v_{i_1}), (v_{i_1}, v_{i_2}), \dots, (v_{i_k}, v_j) \in E(G)$.



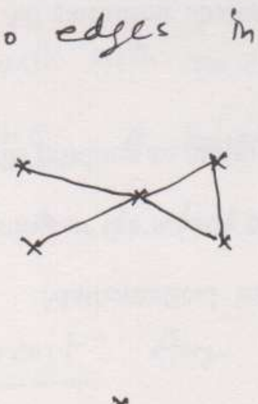
Connected graph:

A graph $G=(V,E)$ is connected if for any two vertices $v_i, v_j \in V(G)$, there exists a path from v_i to v_j.

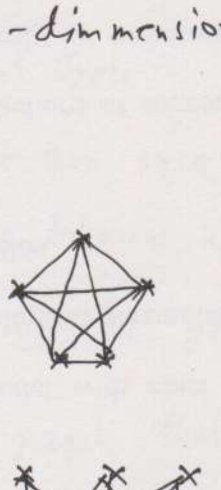
A graph which is not connected is called a disconnected graph.

Planar Graph:

A planar Graph is a graph whose vertices and edges can be drawn in a planee (2-dimmension) such that no two edges intersect.

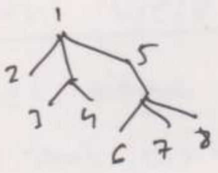


Planar

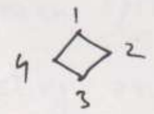


Non-planar

Tree: A tree is a graph in which any two vertices are connected by ~~at most~~ exactly one path.



Tree

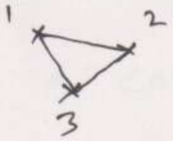


not a tree

A disjoint union of trees is called a Forest.

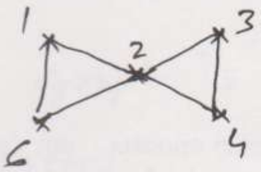
Circuit: In a graph, a circuit is a closed trial. Thus a circuit starts and ends at the same vertex but no edge is repeated.

A circuit is called a cycle if in this circuit no vertex except for the first and the last is repeated.



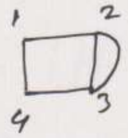
(12)-(23)-(31)

Circuit & Cycle



(12)-(23)-(34)-(42)-(26)-(61)

Circuit but not Cycle



(12)-(23)-(32),
(41)-(34)-(23)

Neither Circuit nor Circle.

Trial: A walk with no edge is travelled more than once is a Trial.

Euler line: A trial in a graph is known as an Euler line if it includes every edge of the graph.

Euler Circuit: Any Circuit in a graph that contains all the edges is called an Euler Circuit.

Euler Graph: A graph which has an Euler Circuit is called an Euler graph.

* A graph that has an Eularian trial but not an Eularian Circuit is called semi-Eularian.

<u>Hamiltonian Cycle:</u> A Hamiltonian Cycle is a cycle that visits each ~~edge~~ vertex exactly once.

* A graph that contains a Hamiltonian cycle is called a Hamiltonian graph.

<u>Theorem:</u> A complete graph with more than two vertices is Hamiltonian.

<u>Theorem:</u> Every cycle graph is Hamiltonian.

<u>Dirac's Theorem (1952):</u> A simple graph with n vertices $(n \geq 3)$ is Hamiltonian if every vertex has degree $\frac{n}{2}$ or greater.

<u>Theorem:</u> Any connected graph G will be an Euler graph iff all the vertices of G are of even degree.

<u>Theorem:</u> A connected graph is Eulerian iff it can be decomposed into cycles.

Travelling Salesman Problem:

Starting from a fixed vertex one has to visit every other vertex only once and finally come back to the starting vertex.

Theorem: The Travelling salesman problem can be solved by finding $\frac{1}{2}(n-1)!$ Hamiltoniah circuits.

Definition (Shortest Path):

There may be several paths between two vertices in a graph. The path that contains the minimum number of edges is called a shortest path between the said to vertices. Note that the shortest path may not be unique.

* Dijkstra's Algorithm and Warshall's Algorithm are being used to find a shortest path between two vertices in a graph.