Partial Differential Equation

An equation which contains an unknown function of two or more variable and its partial derivatives with respect to this variable, is known as Partial Differential Equation.

The order of a partial differential equation is the order of the highest derivative present in the equation.

Let us consider the general form of a second order, linear partial differential equation as →

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G.$$

if $G = 0$ → Homogeneous Partial Differential Equation

if $G \neq 0$ → Inhomogeneous Partial Differential Equation.

Where A, B, C, D, E are called co-efficient of Partial Differential Equation.

Depending upon the nature of the solution of this differential equation, the partial differential equation can be specified into 3 parts as →

when $B^2 - 4AC$ = Negative → Elliptic

= Positive → Hyperbolic

= 0 → Parabolic

Examples:

(i) $3\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x \partial y} - 1.5\frac{\partial^2 z}{\partial y^2} = 0$

$B^2 - 4AC = (-5)^2 - 4(3)(-1.5)$

$= 25 + 18 = 43$

∴ This Partial Differential Equation is the Hyperbolic partial differential equation.

(ii) $\frac{\partial^2 \psi}{\partial t^2} = \frac{1}{5^2}\frac{\partial^2 \psi}{\partial x^2}$

↳ $\frac{\partial^2 \psi}{\partial t^2} - 5^2\frac{\partial^2 \psi}{\partial x^2} = 0$

∴ $A = 1,\ B = 0,\ C = -5^2$

∴ $B^2 - 4AC = 0 - 4(1)(-25)$

$= 100$

∴ This differential equation is the hyperbolic partial differential equation.

Some Important Partial Differential Equation:

1. Wave Equation:

$$\nabla^2 \phi = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2}$$

or, $$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = 0$$

i.e; $\Box^2 \phi = 0$ Where, $\Box^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ is called D'Alembertian

↓

a four dimensional analogue of Laplacian in Minkowski space.

$\phi \xrightarrow{\text{represents}}$

- ✓ displacement of a tightly stretched string
- ✓ deflection of a stretched membrane
- ✓ current or potential along an electrical transmission line

2. Heat Flow Equation:

The time dependent heat flow equation is $\nabla^2 \phi = \frac{1}{h^2}\frac{\partial \phi}{\partial t}$

$h^2 \rightarrow$ Constant (diffusivity)

$\phi \rightarrow$ non-steady state temperature with no heat source or it may be the concentration of a diffusing material.

3. Laplace's Equation:

$$\boxed{\nabla^2 \phi = 0}$$

most important and most commonly used.

- ✓ gravitational potential in regions containing no matter
- ✓ electrostatic potential in a uniform dielectric
- ✓ magnetic potential in free space
- ✓ electric potential in the theory of the steady flow of electric currents
- ✓ temperature in the theory of thermal equilibrium
- ✓ velocity potential → a points of a homogeneous liquid moving irrotationally

4. Poisson's Equation:

$$\boxed{\nabla^2 \phi = \rho}$$

$\rho \rightarrow$ source density
→ function of position coordinates

5. Helmholtz Differential Equation:

$$\nabla^2 \phi \pm k^2 \phi = 0$$

$\phi \rightarrow$ time independent part of the solution of either the diffusive or the wave equation

6. Schrödinger's Equation:

The time-independent form: $\nabla^2 \psi + \frac{2m}{\hbar^2}(E-V)\psi = 0$

Time-dependent form: $\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi = i\hbar \frac{\partial \psi}{\partial t}$

Solution of Partial Differential Equation:

Method of Separation of Variables

One-dimensional wave equation $\rightarrow$ $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ — (i)

In this method, let $u = X(x) f(t)$

where X is a function of x and f is a function of t

substituting this in equ^n (i), we get $\rightarrow$

$$f(t)\frac{d^2 X}{dx^2} = \frac{1}{v^2} X(x) \frac{d^2 f}{dt^2} \quad [\text{where } c = v = \text{velocity}]$$

Dividing both side by $X(x) f(t) \rightarrow$

$$\frac{1}{X(x)} \cdot \frac{d^2 X}{dx^2} = \frac{1}{v^2}\frac{1}{f(t)}\frac{d^2 f}{dt^2}$$

Here left hand side is function of 'x', but right hand side is a function of 't', which are independent of each other.

Hence, we must conclude both side must be equal to some constant.

$$= -\lambda^2 \text{ (constant, say)}$$

$$\therefore \frac{1}{X}\frac{d^2 X}{dx^2} = -\lambda^2 \quad \text{or,} \quad \frac{d^2 X}{dx^2} + \lambda^2 X = 0 \text{ —— (ii)}$$

and $$\frac{1}{f(t)} \cdot \frac{1}{v^2}\frac{d^2 f}{dt^2} = -\lambda^2 \quad \text{or,} \quad \frac{d^2 f}{dt^2} + \lambda^2 v^2 f(t) = 0 \text{ —— (iii)}$$

These two are familiar differential equation of S.H.M.

So, its solution will be given by $\longrightarrow$

$$\left.\begin{aligned} X(x) &= A \sin \lambda(x) + B \cos \lambda x \\ f(t) &= C \sin \lambda vt + D \cos \lambda vt \end{aligned}\right\}$$

Thus, the solution of 1-D wave equation is $\longrightarrow$

$$u = (A \sin \lambda x + B \cos \lambda x) \cdot (C \sin \lambda vt + D \cos \lambda vt)$$

Find the solution of this equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$ subject to the condition

$$u(0,t) = 0 \quad \text{--- (i)}$$
$$u(l,t) = 0 \quad \text{--- (ii)}$$
$$u(x,0) = f(x) \quad \text{--- (iii)}$$
$$\frac{\partial u}{\partial t}(x,0) = 0 \quad \text{--- (iv)}$$

We have, $u = (A \sin \lambda x + B \cos \lambda x)(C \sin \lambda ct + D \cos \lambda ct)$

from condition (i), we get $\longrightarrow$ $0 = B(C \sin \lambda ct + D \cos \lambda ct)$

$$\therefore B = 0$$

$$\therefore u = A \sin \lambda x \,(C \sin \lambda ct + D \cos \lambda ct)$$
$$= \sin \lambda x \left(\frac{C}{A} \sin \lambda ct + \frac{D}{A} \cos \lambda ct\right)$$
$$= \sin \lambda x \,(a \sin \lambda ct + b \cos \lambda ct)$$

where, $\frac{C}{A} = a$ and $\frac{D}{A} = b$ = const.

from condition (ii), we get $\longrightarrow$

$$0 = \sin \lambda l \,(a \sin \lambda ct + b \cos \lambda ct)$$

$$\therefore \lambda l = n\pi$$

$$\therefore \lambda = \frac{n\pi}{l} \qquad n = 1, 2, 3, \ldots$$

$$\therefore u = \sin \frac{n\pi}{l} x \left(a \sin \frac{n\pi c}{l} t + b \cos \frac{n\pi ct}{l}\right)$$

$$\therefore u = \sum_{n=1}^{\infty} \sin \frac{n\pi}{l} x \left(a_n \sin \frac{n\pi ct}{l} + b_n \cos \frac{n\pi ct}{l}\right)$$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} \left\{ \frac{n\pi c}{l} a_n \cos \frac{n\pi ct}{l} - \frac{n\pi c}{l} b_n \sin \frac{n\pi c}{l} \right\}$$

from condition (iv), we get →

$$0 = \sum \frac{n\pi c}{l}\left(a_n \sin\frac{n\pi x}{l}\right)$$

$$\therefore a_n = 0$$

$$\therefore u = \sum_{n=1}^{\infty} b_n \cdot \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l}$$

from condition (iii), we have →

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

Which is the fourier series expansion of an odd function within the limit $-l$ to $+l$.

$$\therefore b_n = \frac{2}{l}\int_0^l f(x) \cdot \sin\frac{n\pi x}{l} \cdot dx$$

Thus the solution of the above equation with that condition is given by →

$$\boxed{u = \sum_{n=1}^{\infty}\left\{\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx\right\} \cdot \sin\frac{n\pi x}{l} \cdot \cos\frac{n\pi ct}{l}}$$

find the solution of this equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \cdot \frac{\partial^2 u}{\partial t^2}$ subject to the conditions →

$$u(0,t) = 0 \quad \text{—(i)}$$

$$u(l,t) = 0 \quad \text{—(ii)}$$

$$u(x,0) = 0 \quad \text{—(iii)}$$

$$\frac{\partial u}{\partial t}(x,0) = f(x) \quad \text{—(iv)}$$

We have, $u = (A\sin\lambda x + B\cos\lambda x)(C\sin\lambda ct + D\cos\lambda ct)$

from condition (i), we get → $0 = B(C\sin\lambda ct + D\cos\lambda ct)$

$$\therefore B = 0$$

$$\therefore u = A\sin\lambda x\,(C\sin\lambda ct + D\cos\lambda ct)$$

$$= \sin\lambda x\left(\frac{C}{A}\sin\lambda ct + \frac{D}{A}\cos\lambda ct\right)$$

$$= \sin\lambda x\,(a\sin\lambda ct + b\cos\lambda ct)$$

$\left.\begin{array}{l}\frac{C}{A} = a \\ D/A = b\end{array}\right\} = \text{Constant.}$

from condition (ii), we have →

$$0 = \sin \lambda l \left(a \sin \lambda ct + b \cos \lambda ct\right)$$

$$\therefore \lambda l = n\pi$$

$$\therefore \lambda = \frac{n\pi}{l} \quad \text{where } n = 1, 2, 3, \ldots$$

$$\therefore u = \sin \frac{n\pi x}{l} \left(a \sin \frac{n\pi ct}{l} + b \cos \frac{n\pi ct}{l}\right)$$

$$\therefore u = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} \left\{a_n \sin \frac{n\pi ct}{l} + b_n \cos \frac{n\pi ct}{l}\right\}$$

from condition (iii), we have →

$$0 = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} \cdot b_n$$

$$\therefore \boxed{b_n = 0}$$

$$\therefore u = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \sin \frac{n\pi ct}{l}$$

home work
↓

Q. Solve the differential equation $2x\frac{\partial u}{\partial x} - 3y\frac{\partial u}{\partial y} = 0$ ——— (i)

Here the function u depends on two variables x and y. Let X be function of x and Y the function of y, so we can write →

$$u(x,y) = X(x)\,Y(y) \quad \text{——— (ii)}$$

Equ^n (i) becomes

$$2x\frac{\partial}{\partial x}(XY) - 3y\frac{\partial}{\partial y}(XY) = 0$$

$$\Rightarrow 2x\cdot Y\frac{\partial X}{\partial x} - 3y\,X\frac{\partial Y}{\partial y} = 0$$

Dividing by XY, we get →

$$\frac{2x}{X}\frac{\partial X}{\partial x} - \frac{3y}{Y}\frac{\partial Y}{\partial y} = 0 \Rightarrow \frac{2x}{X}\frac{\partial X}{\partial x} = \frac{3y}{Y}\frac{\partial Y}{\partial y} \quad \text{——— (iii)}$$

In this equation L.H.S. is function of x only, while R.H.S is function of y only. If this equation is satisfied then each side must be equal to same constant (say k), then →

$$\underbrace{\frac{2x}{X}\frac{\partial X}{\partial x} = k}_{\text{Variable } x} \quad \text{and} \quad \underbrace{\frac{3y}{Y}\frac{\partial Y}{\partial y} = k}_{\text{variable } y}$$

$$2x\frac{\partial X}{\partial x} = kX \quad \text{——— (iv)} \qquad \text{and} \qquad 3y\frac{\partial Y}{\partial y} = kY \quad \text{——— (v)}$$

$$\frac{\partial X}{X} = \frac{k}{2}\frac{\partial x}{x} \qquad\qquad \frac{\partial Y}{Y} = \frac{k}{3}\frac{\partial y}{y}$$

$\therefore \log_e X = \frac{k}{2}\log_e x + \log C_1$

$= \log_e(x^{k/2}) + \log_e C_1$

$\therefore X = C_1 x^{k/2}$

$\therefore \log_e Y = \frac{k}{3}\log_e y + \log C_2$

$= \log_e y^{k/3} + \log_e C_2$

$\therefore Y = C_2 y^{k/3}$

$\therefore u = XY = C_1 x^{k/2}\, C_2 y^{k/3}$

$= C x^{k/2} y^{k/3}$ $\quad \boxed{C = C_1 C_2}$

The given differential equation is $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ if $u(x,0) = \sin \pi x$

Let $u(x,t) = X(x)T(t)$

$$\therefore \frac{\partial (XT)}{\partial t} = \frac{\partial^2 (XT)}{\partial x^2} \Rightarrow X\frac{\partial T}{\partial t} = T\frac{\partial^2 X}{\partial x^2}$$

Dividing by XT, we get →

$$\frac{1}{T}\frac{\partial T}{\partial t} = \frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -p^2 \text{ (say)}$$

$$\therefore \frac{\partial T}{T} = -p^2 \partial t \Rightarrow \log_e T = -p^2 t + \log_e A \Rightarrow \boxed{T = Ae^{-p^2 t}}$$

and $\frac{\partial^2 X}{\partial x^2} + p^2 X = 0 \Rightarrow X = (C_1 \cos px + C_2 \sin px)$

$$\therefore u(x,t) = XT = (C_1 \cos px + C_2 \sin px) \cdot Ae^{-p^2 t}$$

$$\therefore u(x,0) = (C_1 \cos px + C_2 \sin px) A$$

Given $u(x,0) = \sin \pi x$;

therefore, $\sin \pi x = AC_1 \cos px + AC_2 \sin px$

Comparing coefficients of $\sin px$ and $\cos px$ on both sides.

$AC_1 = 0$

$AC_2 = 1$

$p = \pi$

$$\therefore u = e^{-p^2 t} \sin \pi x$$

Solution of Laplace's Equation in Cartesian Coordinates:

(Method of Separation of variables)

The Laplace's equation in Cartesian coordinate is →

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \text{——(i)}$$

Obviously u is the function of (x, y, z).

Using the method of separation of variables u may be expressed as →

$$u = X(x)\, Y(y)\, Z(z) \quad \text{——(ii)}$$

Substituting (ii) in (i), we get →

$$YZ \frac{\partial^2 X}{\partial x^2} + ZX \frac{\partial^2 Y}{\partial y^2} + XY \frac{\partial^2 Z}{\partial z^2} = 0 \quad \text{——(iii)}$$

Dividing throughout by XYZ, we get →

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = 0$$

$$\text{or, } \frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} - \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} \quad \text{——(iv)}$$

In this equation L.H.S is the function of x only while right hand side is the function of y and z only; therefore each side must be equal to the same constant say, (k_1^2)

i.e;
$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = k_1^2 \quad \text{or} \quad \frac{\partial^2 X}{\partial x^2} - k_1^2 X = 0 \quad \text{——(v)}$$

and
$$-\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} - \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = k_1^2$$

$$\text{or, } \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} - k_1^2$$

In this equation LHS is the function of y only; while RHS is the function of z only, therefore each side must be equal to the same constant k_2^2 (say) i.e,

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = k_2^2 \quad \text{or,} \quad \frac{\partial^2 Y}{\partial y^2} - k_2^2 Y = 0 \quad \text{——(vi)}$$

$$\text{and } -\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} - k_1^2 = k_2^2 \quad \text{or, } \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = -(k_1^2 + k_2^2) = -k_3^2 \text{ (say)}$$

Thus $\frac{\partial^2 Z}{\partial z^2} - k_3^2 Z = 0$ ——— (vii)

and $-(k_1^2 + k_2^2) = k_3^2$ or, $k_1^2 + k_2^2 + k_3^2 = 0$ ——— (viii)

The solutions of equations (v), (vi) and (vii) may be expressed as

$$X = A e^{k_1 x}, \quad Y = B e^{k_2 y}, \quad Z = C e^{k_3 z} \qquad \text{——— (ix)}$$

The general solution of (i) thus becomes

$$u = XYZ = ABC\, e^{k_1 x} e^{k_2 y} e^{k_3 z} = ABC\, e^{k_1 x + k_2 y + k_3 z} \qquad \text{——— (x)}$$

Here k_1, k_2, k_3 are arbitrary constants (positive or negative) related by (viii).
Equ^n (x) represents solution for a particular choice of k_1, k_2, k_3. As they may possess infinite number of values, therefore the complete solution of (i) would be of the form →

$$u = \sum_{k_1, k_2, k_3} N_{k_1 k_2 k_3}\, e^{k_1 x + k_2 y + k_3 z} \qquad \text{——— (xi)}$$

Where $N_{k_1 k_2 k_3} = ABC$ is an arbitrary constant and may be evaluated by initial conditions of the specified problem.

$$\therefore u = \sum_{k} N_k\, e^{\vec{k}\cdot\vec{r}}, \quad |k| \neq 0$$

VII Solution of Laplace's Equation in General Cylindrical coordinates:

(General Cylindrical Harmonics)

The Laplace's equation $\nabla^2 u = 0$

in $\underbrace{\text{cylindrical coordinate}}_{(r,\theta,z)}$ is $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$ ——— (i)

In this equation u is the function of r, θ and z, therefore by the method of separation of variables u may be written as

$$u(r,\theta,z) = R(r)\, \Theta(\theta)\, Z(z) \qquad \text{——— (ii)}$$

Substituting value of u in equation (i) and dividing by $R\Theta Z$, we get →

$$\frac{1}{rR}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)+\frac{1}{r^2\Theta}\frac{\partial^2\Theta}{\partial\theta^2}+\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}=0$$

$$\text{or,}\quad \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}=-\frac{1}{rR}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)-\frac{1}{r^2\Theta}\frac{\partial^2\Theta}{\partial\theta^2} \quad\text{———— (iii)}$$

In this equation LHS is the function of z alone while RHS is the function of r and θ and is independent of z, therefore each side must be equal to the same constant k^2 (say), i.e;

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}=k^2 \quad\text{or,}\quad \frac{\partial^2 Z}{\partial z^2}-k^2 Z=0 \quad\text{———— (iv)}$$

$$\text{and}\quad -\frac{1}{rR}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)-\frac{1}{r^2\Theta}\frac{\partial^2\Theta}{\partial\theta^2}=k^2 \quad\text{———— (v)}$$

Multiplying equ^n (v) throughout by r^2, we get $\rightarrow$

$$-\frac{r^2}{R}\frac{\partial^2 R}{\partial r^2}-\frac{r}{R}\frac{\partial R}{\partial r}-\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial\theta^2}=k^2r^2$$

$$\text{or,}\quad +\frac{r^2}{R}\frac{\partial^2 R}{\partial r^2}+\frac{r}{R}\frac{\partial R}{\partial r}+k^2r^2=-\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial\theta^2} \quad\text{———— (vi)}$$

Again the LHS is the function of r only; while RHS is the function of θ only; therefore each side must be equal to the same constant m^2 (say), i.e.

$$-\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial\theta^2}=m^2$$

$$\text{or,}\quad \frac{\partial^2\Theta}{\partial\theta^2}+m^2\Theta=0 \quad\text{———— (vii)}$$

$$\text{and}\quad \frac{r^2}{R}\frac{\partial^2 R}{\partial r^2}+\frac{r}{R}\frac{\partial R}{\partial r}+k^2r^2=m^2$$

$$\text{or,}\quad r^2\frac{\partial^2 R}{\partial r^2}+r\frac{\partial R}{\partial r}+(k^2r^2-m^2)R=0 \quad\text{———— (viii)}$$

The solution of equ^n (iv) is given as $Z=C_1e^{kz}+C_2e^{-kz}$ ———— (ix)

The solution of equ^n (vii) may be expressed as $\Theta=C_3\cos m\theta+C_4\sin m\theta$ ———— (x)

In order to solve equ^n (viii), let us substitute

$$kr=x \quad\text{———— (xi)}$$

then equation (viii) takes the form

$$x^2 \frac{\partial^2 R}{\partial x^2} + x \frac{\partial R}{\partial x} + (x^2 - m^2) R = 0 \quad \text{——— (xii)}$$

This is <u>Bessel's differential equation.</u>

Its solution is given by

$$R = C_5 J_m(x) + C_6 J_{-m}(x)$$

$$= C_5 J_m(kr) + C_6 J_{-m}(kr)$$

for m as a fraction

$$\text{or, } R = C_5 J_m(kr) + C_6 Y_m(kr)$$

for m an integer or in general

Thus the general solution of Laplace's equation in cylindrical coordinates (r, θ, z) is given by

$$u(r, \theta, z) = R(r)\, \Theta(\theta)\, Z(z)$$

$$= \left[C_1 e^{kz} + C_2 e^{-kz}\right]\left[C_3 \cos m\theta + C_4 \sin m\theta\right]\left[C_5 J_m(kr) + C_6 Y_m(kr)\right]$$

↳ These solutions of Laplace's equation are called general <u>cylindrical harmonics.</u>

Solution of Laplace's Equation in Spherical Polar Coordinates:

(Spherical Harmonics)

The Laplace's equation $\nabla^2 u = 0$ in spherical polar coordinates (r, θ, ϕ) takes the form :

$$\frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 u}{\partial \phi^2} = 0$$

$$\Rightarrow \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{\sin^2\theta} \frac{\partial^2 u}{\partial \phi^2} = 0 \quad \text{——— (i)}$$

from this equation it is obvious that u is the function of (r, θ, ϕ); therefore by the method of separation of variables its solution may be expressed as →

$$u(r, \theta, \phi) = R \Theta \Phi \quad \text{——— (ii)}$$

Substituting (ii) in (i), we get →

$$\frac{\partial}{\partial r}\left\{r^2 \frac{\partial (R\Theta\Phi)}{\partial r}\right\} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left\{\sin\theta \frac{\partial(R\Theta\Phi)}{\partial\theta}\right\} + \frac{1}{\sin^2\theta}\frac{\partial^2(R\Theta\Phi)}{\partial\phi^2} = 0$$

Dividing by $R\Theta\Phi$, we get →

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + \frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \frac{1}{\Phi\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0$$

Multiplying througout by $\sin^2\theta$, we get →

$$\frac{\sin^2\theta}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{\sin\theta}{\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} = 0$$

$$\Rightarrow \frac{\sin^2\theta}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{\sin\theta}{\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) = -\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} \quad \text{——— (iii)}$$

In this equation LHS is the function of r and θ only, while RHS is a function of ϕ only, therefore for the validity of this equation, each side must be equal to same constant m^2 (say), so that

$$\frac{\sin^2\theta}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{\sin\theta}{\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) = m^2 \quad \text{——— (iv)}$$

and $$-\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} = m^2 \Rightarrow \frac{\partial^2\Phi}{\partial\phi^2} + m^2\Phi = 0 \quad \text{——— (v)}$$

Now dividing equation (iv) by $\sin^2\theta$, we get →

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) = \frac{m^2}{\sin^2\theta}$$

$$\Rightarrow \underbrace{\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)}_{\text{function of } r} = \underbrace{-\frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \frac{m^2}{\sin^2\theta}}_{\text{function of } \theta}$$

$$\therefore \frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = n(n+1) \quad \text{(Say)}$$

$$\Rightarrow r^2\frac{\partial^2 R}{\partial r^2} + 2r\frac{\partial R}{\partial r} - n(n+1)R = 0 \quad \text{——— (vi)}$$

and $$-\frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \frac{m^2}{\sin^2\theta} = n(n+1)$$

$$\Rightarrow \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \left[n(n+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0 \quad \text{——— (vii)}$$

The solution of equation (vi) is $\rightarrow$

$$R = Ar^{n} + Br^{-n-1} \quad \text{——— (viii)}$$

In equation (vii), we put $x = \cos\theta$ and transform the variable θ in x.

$$\frac{\partial \Theta}{\partial \theta} = \frac{\partial \Theta}{\partial x} \cdot \frac{\partial x}{\partial \theta} = -\sin\theta \frac{\partial \Theta}{\partial x}$$, so equ^n (vii) becomes

$$-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin^2\theta \frac{\partial \Theta}{\partial x}\right] + \left[n(n+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0$$

$$\Rightarrow \frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial \Theta}{\partial x}\right] + \left[n(n+1) - \frac{m^2}{(1-x^2)}\right]\Theta = 0$$

$$\Rightarrow (1-x^2)\frac{\partial^2 \Theta}{\partial x^2} - 2x\frac{\partial \Theta}{\partial x} + \left[n(n+1) - \frac{m^2}{(1-x^2)}\right]\Theta = 0$$

——— (ix)

This is associated Lagendre's equation, its solution is $\rightarrow$

$$\Theta = C P_n^m(x) + D Q_n^m(x)$$

or, $$\Theta = C P_n^m(\cos\theta) + D Q_n^m(\cos\theta) \quad \text{——— (x)}$$

The solution of equ^n (v) is $\rightarrow$

$$\Phi = E\cos(m\phi) + F\sin(m\phi) \quad \text{——— (xi)}$$

Therefore the solution of Laplace's equation will be $\rightarrow$

$$u = R\Theta\Phi = \left(Ar^{n} + Br^{-n-1}\right)\left[C P_n^m(\cos\theta) + D Q_n^m(\cos\theta)\right]$$
$$\left(E\cos m\phi + F\sin m\phi\right) \quad \text{——— (xii)}$$

Diffusion Equation or, Fourier Equation of Heat Flow:

We know that heat flows from regions of higher temperature to those of lower temperature and experiments show that the rate of heat flow is proportional to the temperature gradient.

This implies that the velocity v of heat flow in a body is of the form $\rightarrow$ $v = -K \,\text{grad}\, u$ ——— (i)

where K is a constant called the thermal conductivity of the body and

$u(x,y,z,t)$ is the temperature at position (x,y,z) and time t.

Now consider a region of volume V bounded by surface S of the body. Consider a small element ds of the surface S, the direction of this surface element is along the outward drawn normal.

If $\hat{n}$ is unit vecter along this outward drawn normal, then the amount of heat leaving the surface S in unit time is given by →

$$H = \iint_S \vec{v} \cdot \vec{ds} = \iint_S \vec{v} \cdot \hat{n}\, ds \quad \text{———— (ii)}$$

Using Gauss divergence theorem for vector $\vec{A}$

i.e. $$\iint_S \vec{A} \cdot \vec{ds} = \iiint_V \operatorname{div} \vec{A}\, d\vec{v}$$

to convert surface integral into volume integral, we get →

$$H = \iint_S \vec{v} \cdot \hat{n}\, ds = \iiint_V \operatorname{div}(V)\, dV$$

$$= \iiint_V \operatorname{div}(-K \operatorname{grad} u)\, dV = -K \iiint_V \operatorname{div} \operatorname{grad} u\, dV$$

$$= -K \iiint_V \vec{\nabla} \cdot \vec{\nabla} u\, dV = -K \iiint_V \nabla^2 u\, dV \quad \text{———— (iii)}$$

Now if ρ denotes the density of body, c its specific heat, then the total amount of heat absorbed by volume V is given by

$$Q = \iiint_V (\rho\, dV) \cdot cu = \iiint_V \rho\, cu\, dV \quad \text{———— (iv)}$$

Therefore the rate of decrease of heat is

$$H = -\frac{\partial Q}{\partial t} = -\iiint_V \rho c \frac{\partial u}{\partial t}\, dV \quad \text{———— (v)}$$

This rate of decrease of heat must be equal to the amount of heat leaving the volume V per unit time ~~is~~, i.e.,

$$-K \iiint_V \nabla^2 u\, dV = -\iiint_V \rho c \frac{\partial u}{\partial t}\, dV \quad \text{———— } \cancel{(vi)}$$

or, $\iiint_V \left(K\nabla^2 u - \rho c \frac{\partial u}{\partial t} \right) dv = 0$ ——— (vi)

As volume element dv is arbitrary, therefore equ^n (vi) holds only if the integrand is zero, i.e:

$$K\nabla^2 u - \rho c \frac{\partial u}{\partial t} = 0$$

or, $\nabla^2 u = \frac{\rho c}{K} \frac{\partial u}{\partial t}$

i.e, $\boxed{\nabla^2 u = \frac{1}{h^2} \frac{\partial u}{\partial t}}$ ——— (vii) → 3-D Diffusion Equation. or, Fourier Equation of Heat Flow.

Where $h^2 = \frac{K}{\rho c}$ is constant and is called the diffusivity of the substance.