P-1

Notes by
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Power Series:

A power series (in one variable) is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

variable

is a constant (center of the series)

co-efficients

If $c=0$, the series becomes

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

also called Maclaurin series.

Example: . Take $a_n = \frac{1}{n}$ $c=5$ ....

- Any polynomial is a power series.

- Geometric series: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + \cdots$, $|x|<1$.

- Exponential function: $\sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2!} + \cdots$, $\forall x \in \mathbb{R}$.
$= e^x$

- Sine function: $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$, $\forall x \in \mathbb{R}$.

Note: . Negative powers are not allowed in a power series. Those are called Laurent series.

- Fractional powers are also not allowed. Those are called Puiseux series.
- co-efficients can not be a functions of x.

Radius of Convergence:

$\{ x : \sum_{n=0}^{\infty} a_n (x-c)^n \text{ converges} \}$ is the interval of convergence.

FACT: The power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ either

(1) converge at $x=c$ and diverges elsewhere,

(2) converges absolutely $\forall x$, or

(3) converges absolutely for $|x-c| < R$ and diverges for $|x-c| > R$, where $0 < R < \infty$.

The endpoints $x = c \pm R$ must be tested separately for convergence.

Note: The number R is called the Radius of Convergence.

In case (1), $R = 0$.

In case (2), $R = \infty$.

Tests for convergence:

Ratio test: Suppose $a_n \neq 0$ for all sufficiently large n and the limit $R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|$

exists or diverges to infinity. Then the power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ has radius of converge R.

interval of con. is $(c-R, c+R)$.

Example: Consider: $\sum_{n=1}^{\infty} \frac{4^n n^3}{n!}$, $\sum_{n=0}^{\infty} \frac{(-2)^{2n}}{3^n (n+2)!}$,

$\sum_{n=1}^{\infty} \frac{(2n+3)!}{n^2}$, $\sum_{n=0}^{\infty} \frac{3n+6}{n+2}$, $\sum_{n=1}^{\infty} \frac{4^n + n}{(n+1)!}$, Cauchy- Hadamard test:

The radius of conv. R of the power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ is given by $R = \limsup_{n\to\infty} |a_n|^{1/n}$.

Example: Consider $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n$, $\sum_{n=0}^{\infty} (-1)^n x^{2^n}$,

$$\downarrow$$

$$|a_n|^{1/n} = \begin{cases} 1 & \text{if } n = 2^k \\ 0 & \text{if } n \neq 2^k \end{cases}.$$

Differentiation of Power series:

In general one can not differentiate a uniformly convergent series. But differentiable within interval of conv.

Theorem: Let the series $\sum_{n=1}^{\infty} a_n (x-c)^n$ conv. $\forall\ |x-c| < R$ and define $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ $(\forall\ |x-c| < R)$.

(1) Then the series converges uniformly on $[c-R+\varepsilon, c+R-\varepsilon]$ for any $\varepsilon > 0$.

(2) The function $f(x)$ is continuous & differentiable in $(c-R+\varepsilon, c+R+\varepsilon)$ with $f'(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1}$, $(|x-c| < R)$.

Corollary: A power series is infinitely differentiable within interval of convergence.

Taylor Series: Let $f : (c-\delta, c+\delta) \to \mathbb{R}$ be infinitely differentiable. Then, $\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n$ is the Taylor series of f. at $x=c$

FACT: Let $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ conv. in $|x-c| < R$. for some R,

Then f has derivatives of all order.

$$\therefore f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1) \cdots (n-k+1) a_n (x-c)^{n-k}.$$

In particular, $f^{(k)}(c) = k!\, a_k$ i.e. $a_k = \dfrac{f^{(k)}(c)}{k!}$

$\forall\, k = 0, 1, 2, \ldots$

Uniqueness of Power series:

Suppose the power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n \quad \text{and} \quad f(x) = \sum_{n=0}^{\infty} b_n (x-c)^n.$$

both with radius of conv. $R > 0$, then $a_n = b_n \ \forall n$.

Term by Term integration of Power series:

FACT: Let $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ for $|x-c| < R$.

Then, (1) f has anti derivative $F(x)$ given by

$$F(x) = \int f(x)\,dx = \sum_{n=0}^{\infty} \frac{a_n (x-c)^{n+1}}{n+1} + K \quad \text{for } |x-c| < R.$$

constant

(2) For $[\alpha, \beta] \subseteq (c-R, c+R)$,

$$\int_{\alpha}^{\beta} f(x)\,dx = \sum_{n=0}^{\infty} \left[\int_{\alpha}^{\beta} a_n (x-c)^n\,dx \right].$$

-5

Representing functions as power series:

Power series for $\tan^{-1}(x)$:

Recall that, $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$

$\therefore \int \frac{dx}{1+x^2} = \tan^{-1}x$.

Now $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n} \;\forall\, \underline{|-x^2|<1}$

$\therefore \int_0^x \frac{dx}{1+t^2} = \sum_{n=0}^{\infty}\int_0^x (-1)^n t^{2n}\,dt = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$

The series has center of conv. 0, conv. for $|-x^2|<1$

i.e. $|x|<1$.

Abel's Theorem:

Let $G(x) = \sum_{k=0}^{\infty} a_k x^k$, $a_k \in \mathbb{R}$, radius of conv. 1.

Suppose $\sum_{k=0}^{\infty} a_k$ converges. Then, $G(x)$ is continuous at $x=1$

i.e. $\lim_{x\to 1^-} G(x) = G(1) = \sum_{k=0}^{\infty} a_k$.

Remark: 1 may be replaced by any R.

Converse of Abel's theorem is not true!

Let $g(x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$.

$g(x)$ has a limit at $x\to 1^-$. i.e. $\lim_{x\to 1^-} g(x) = \frac{1}{2}$.

But the power series does not converge at $x=1$.

P-6

Weierstrass Approximation Theorem:

Theorem: Suppose f is a continuous real-valued function defined on the real interval $[a,b]$. For every $\varepsilon > 0$, $\exists$ a polynomial $p(x)$ such that $\forall x \in [a,b]$, we have $|f(x) - p(x)| < \varepsilon$, or $\|f - p\| = \sup_{x\in[a,b]} |f(x) - p(x)| < \varepsilon$.

Problem: Suppose f is continuous and

$$\int_0^1 f(x)\, x^n \, dx = 0 \quad \forall\, n = 0, 1, 2, \dots$$

Then show that $f \equiv 0$.